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Computationally Efficient Methods for Estimating Unknown Input Forces on Structural Systems

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***Abstract*—We consider the problem of estimating unknown input forces on structural systems using *only noisy acceleration measurement* data. This is an important task for condition monitoring, for example, to predict fatigue damage in a structure's body or to reduce transmission of vibrations in marine vessels. In this paper, we propose a new idea to estimate an input force with a sinusoidal form by formulating a force identification problem without a direct feed-through system. Consequently, the minimum variance unbiased (MVU) filter can be implemented coupled with a fast Fourier transform algorithm to estimate unknown input forces accurately in real-time. Moreover, when the input force is *completely unknown*, the ensemble sampling method combined with an augmented Kalman filter can be formulated to significantly reduce computation time. Experimental results confirm the effectiveness of our proposed methods and show that the formulations investigated outperform other state-of-the-art methods in terms of computational cost whilst not compromising estimation performance.**

***Index Terms*—Non-linear filtering, Bayes estimations, Mass-spring-damper system, Minimum variance unbiased (MVU) filters, Ensemble Augmented Kalman filters (EnAKF).**

I. Introduction

State estimation of a system plays an important role in condition monitoring the structural health of various mechanical systems in various civil, mechanical or marine engineering applications. Very often, this estimation is implemented by monitoring acoustic noise or structural vibration coming from different parts of the original system. The conventional aim of condition monitoring is in fault detection and predictive maintenance [1], [2]. For example, state estimation can be used to predict fatigue damage in a structure's body [3], or to develop a dynamic hydraulic absorber to reduce vibration transmission in marine vessels [4]. One of the popular methods in state estimation is to model systems directly in the time domain, so-called state-space modelling, by describing the relationship between state variables and measurements via Bayes's rules. In particular, Bayes state estimation (or Bayes filtering) is an online method dealing with the problem of inferring knowledge about the unobserved state of a dynamic system, which changes over time, from a sequence of noisy observations. One of the earliest forms of Bayes filters is the Kalman filter (KF), firstly introduced in 1960 by Kalman [5], wherein the state dynamic and measurement models are linear, and their corresponding variables follow Gaussian distributions. When dynamic and measurement models are somewhat non-linear, two conventional approximate solutions are extended Kalman filter (EKF) [6], [7] and unscented Kalman filter (UKF) [8], [9]. When dynamic and measurement models are extremely non-linear, a particle filter (PF) [10]–[12] is usually implemented.

In structural dynamics, the degrees-of-freedom (DoF) is comprised of the displacements (positions) and velocities that characterise the displacements of the masses related to its initial location. Early efforts of estimating unknown structural states using measurements of displacements and/or velocities have been investigated in [13]–[16]. However, it is *extremely challenging to measure the system's displacements and velocities* directly in order to apply a standard Kalman filter wherein state, and measurement variables are only displacements and velocities. In practice, commonly available measurement data of structural systems are only acceleration data collected at a few DoF mass points using accelerometers [17]. Subsequently, various applications for estimating state variables using Kalman filters with acceleration data only have been proposed [18]–[20].

Besides state estimation, another important aspect of structural monitoring is identifying inputs/parameters of the systems. The EKF, which is based on linearisation of the non-linear dynamics and measurement models, is the most commonly used method in several applications such as parameter identification [21], [22], damage detection [23], [24] or updating model [25] wherein the input forces are known [26]. Other techniques for identifying parameters in highly non-linear systems use the UKF [27] or PF [28], which are computationally more demanding. Detailed comparisons between UKF and PF filters are investigated in [29], which confirms the effectiveness of PF compared to the EKF and UKF for identifying parameters in highly non-linear systems [3]. However, the major drawback of standard PF is its high computational cost; this makes a standard PF formulation infeasible for problems with a high dimensional space such as that with an n-DoF system. Although the modern versions of PF (*e.g.*, particle flow filters [30], [31]) can mitigate the curse of dimensionality problems, an initial assumption about distribution (PF) or mean and covariance (EKF/UKF) of the estimated parameters is required. An incorrect initial estimate will be detrimental to the efforts on estimation outcomes. Moreover, possible measurement outliers can noticeably affect the estimation performance.

The idea of jointly estimating the state and unknown inputs/parameters using the output measurements has emerged recently. The early approaches include i) considering unknown parameters as additional state variables [32], ii) using an interactive multiple model procedure [32], iii) solving the state estimation and model parameters identification problems jointly by a two-step iterative procedure: first, estimate states by assuming model parameters are known completely, then, identify model parameters using the estimated states. See [33] and references therein for more details. The first optimal approach was proposed by Madapusi [34] and Gillijns [35], which can be formulated as a minimum variance unbiased (MVU) filter *without* direct feed-through (transmission) for optimal control applications. This filter is both unbiased and optimal. However, the MVU filter *without* direct feed-through requires the unknown input force to be differentiable (*e.g.*, an input force with a sinusoidal form is differentiable). The unbiased filter *with* direct feed-through is also introduced in [36] by Gillijins and De Moor and is named after its authors as the GDF. A different approach is to augment the state with the unknown input which leads to the so-called augmented Kalman filter (AKF), firstly proposed in [37] for the KF, and later generalised for the EKF in [38]. However, the AKF may lead to untrustworthy estimation if only acceleration measurement data are available [39], which can be alleviated by using dummy displacement measurements. Another approach to jointly estimating state/inputs is to employ the so-called dual Kalman filter (DKF) [3], [40], for a linear state-space model involving an accurate structural model without noise [26]. Although these filters (GDF, AKF, DKF) can estimate the unknown input force accurately, its computational demand is prohibitively high, which prevents its real-time implementation for large and complex n-DoF structural systems.

In this paper, we focus on the problem of estimating an unknown input force parameter for a large n-DoF structural system efficiently and effectively. If the input force has a sinusoidal form, we propose using the MVU filter *without* direct feed-through to estimate the force parameter since MVU filters are fast, optimal and unbiased. When the input force form is unknown, the problem is formulated as a filtering problem *with* direct feed-through. One of the efficient filters for a large

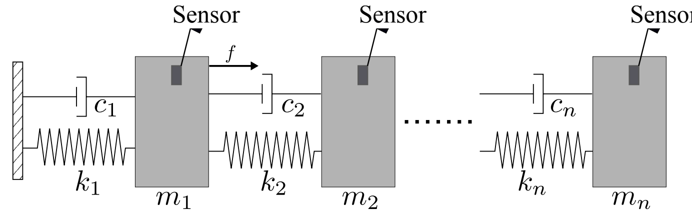

Fig. 1. An n-DoF structural system.

system is the ensemble Kalman filter (EnKF), proposed by Evensen [41]–[43] wherein the estimated states are represented as an ensemble of possible state vectors randomly generated using a Monte Carlo method. The EnKF algorithm does not require any linearisation of the model as in EKF (a first-order Taylor series), and also does not require computing the full error covariance matrix; this leads to noticeable computational cost reductions. Therefore, we adapt the EnKF into the AKF and devise the so-called EnAKF to reduce the computational cost.

II. Problem Formulation

A. Notations

For notational simplicity, lowercase letters denote scalar values (*e.g.*, x), uppercase letters denote vectors (*e.g.*, X), while bold uppercase letters denote matrices (*e.g.*, $\mathbf{X}$). Moreover, if x denotes displacement, then $\dot{x}$ and $\ddot{x}$ denote its corresponding velocity and acceleration, respectively. Further, we use the function $\mathrm{diag}(D, i)$ to represent a $[\mathrm{length}(D) + i] \times [\mathrm{length}(D) + i]$ square matrix, where the vector D is the i^{th} diagonal (relative to the main diagonal, *i.e.*, $i = 0$) of the created matrix. Further, we abbreviate $\mathrm{diag}(D, 0) = \mathrm{diag}(D)$. $(\cdot)^T$ denotes the transpose of the vector or matrix $(\cdot)$.

B. The State-Space Model of an n-DoF Structural System

The dynamic motion model of an n-DoF structural system illustrated in Fig. 1 can be described as:

$$\mathbf{M}\ddot{X}(t) + \mathbf{C}\dot{X}(t) + \mathbf{K}X(t) = F(t), \tag{1}$$

where $\mathbf{M}$ denotes an $n \times n$ system mass matrix, $\mathbf{C}$ denotes an $n \times n$ damping coefficient matrix, $\mathbf{K}$ denotes an $n \times n$ spring stiffness matrix, $F(t)$ denotes the $n \times 1$ input force vector, and $\ddot{X}(t), \dot{X}(t), X(t)$ are the $n \times 1$ vectors that denote the acceleration, velocity and position (*i.e.*, displacement) at time t, respectively. Here, $\mathbf{M} = \mathrm{diag}([m_1, m_2, \ldots, m_n])$, $\ddot{X} = [\ddot{x}_1, \ddot{x}_2, \ldots, \ddot{x}_n]^T$, $\dot{X} = [\dot{x}_1, \dot{x}_2, \ldots, \dot{x}_n]^T$, $X = [x_1, x_2, \ldots, x_n]^T$, $F(t) = S_f f(t)$ where $S_f = [1, 0_{1\times(n-1)}]^T$ with $0_{1\times(n-1)}$ denoting the $1 \times (n-1)$ zero-vector and $f(t)$ is the unknown input force applied to mass m_1.

As in [44], we assume that we can approximate the damping $f^{(c)}$ and spring $f^{(k)}$ forces using the analytic functions of relative position ϵ and velocity $\dot{\epsilon}$, such that

$$f_i^{(c)} = c_i\dot{\epsilon}_i + \tilde{c}_i[\dot{\epsilon}_i]^3, f_i^{(k)} = k_i\epsilon_i + \tilde{k}_i[\epsilon_i]^3, \tag{2}$$

$$\dot{\epsilon}_i = \dot{x}_{i+1} - \dot{x}_i, \epsilon_i = x_{i+1} - x_i, \forall i = 1, \ldots, n, \tag{3}$$

where $c_i, \tilde{c}_i, k_i$ and $\tilde{k}_i$ are constants. Hence, the damping coefficient matrix $\mathbf{C}$ and the stiffness matrix $\mathbf{K}$ can be approximated as followed [44]:

$$\begin{aligned}\mathbf{C} = &\operatorname{diag}([c_1 + \tilde{c}_1(\dot{x}_2 - \dot{x}_1)^2, \ldots, c_n + \tilde{c}_n(\dot{x}_{n+1} - \dot{x}_n)^2]) \\ &- \operatorname{diag}([c_1 + \tilde{c}_1(\dot{x}_2 - \dot{x}_1)^2, \ldots, c_{n-1} + \tilde{c}_{n-1}(\dot{x}_n - \dot{x}_{n-1})^2], 1) \\ &- \operatorname{diag}([c_1 + \tilde{c}_1(\dot{x}_2 - \dot{x}_1)^2, \ldots, c_{n-1} + \tilde{c}_{n-1}(\dot{x}_n - \dot{x}_{n-1})^2], -1)\end{aligned} \tag{4}$$

$$\begin{aligned}\mathbf{K} = &\operatorname{diag}([k_1 + \tilde{k}_1(x_2 - x_1)^2, \ldots, k_n + \tilde{k}_n(x_{n+1} - x_n)^2]) \\ &- \operatorname{diag}([k_1 + \tilde{k}_1(x_2 - x_1)^2, \ldots, k_{n-1} + \tilde{k}_{n-1}(x_n - x_{n-1})^2], 1) \\ &- \operatorname{diag}([k_1 + \tilde{k}_1(x_2 - x_1)^2, \ldots, k_{n-1} + \tilde{k}_{n-1}(x_n - x_{n-1})^2], -1)\end{aligned} \tag{5}$$

where $x_{n+1} = \dot{x}_{n+1} = 0$.

The problem is to estimate the unknown input force $f(t)$ applied to the mass m_1 using only acceleration measurement data.

III. Estimating an unknown input force with a sinusoidal form

In this section, we present a method of estimating an unknown input force with a sinusoidal form using an MVU (minimum variance unbiased) filter by formulating the problem as a linear system *without* direct feed-through of the input force to the measurements.

Assume that the sinusoidal force applied to the mass m_1 is

$$f = F_0 \sin(w_0 t + \phi_0). \tag{6}$$

where F_0 is the force amplitude, w_0 is the force frequency, and ϕ_0 is the force phase offset. Hence, by taking the derivative of the force, *i.e.*,

$$\frac{df}{dt} = F_0 w_0 \cos(w_0 t + \phi_0), \tag{7}$$

and incorporating acceleration as one of the elements of the state vector, *i.e.*, $\mathcal{X} = [X^T, \dot{X}^T, \ddot{X}^T]^T = [X_1^T, X_2^T, X_3^T]^T$, we can discretise the continuous system in (1) into a state-space model, as follows:

$$\begin{aligned}\begin{bmatrix} X_{1,k} \\ X_{2,k} \\ X_{3,k} \end{bmatrix} &= \begin{bmatrix} \mathbf{I}_n & \mathbf{I}_n \triangle t & \mathbf{I}_n \triangle t^2/2 \\ \mathbf{0}_n & \mathbf{I}_n & \mathbf{I}_n \triangle t \\ \mathbf{0}_n & -\dfrac{\mathbf{K}}{\mathbf{M}} \mathbf{I}_n \triangle t & \mathbf{I}_n - \dfrac{\mathbf{C}}{\mathbf{M}} \triangle t \end{bmatrix} \begin{bmatrix} X_{1,k-1} \\ X_{2,k-1} \\ X_{3,k-1} \end{bmatrix} \\ &\quad + \begin{bmatrix} 0_{2n \times 1} \\ S_f \end{bmatrix} \frac{F_0 w_0 \triangle t}{m_1} \cos[w_0(k-1)\triangle t + \phi_0].\end{aligned} \tag{8}$$

where $\mathbf{I}_n$ denotes the $n \times n$ identity matrix, $\mathbf{0}_n$ denotes the $n \times n$ zero matrix, $0_{m \times 1}$ denotes the $m \times 1$ zero-vector, and $\triangle t$ is a discrete measurement time step. Equation (8) can be simply expressed as

$$\mathcal{X}_k = \mathbf{A}\mathcal{X}_{k-1} + G d_{k-1}. \tag{9}$$

Here, $\mathbf{A} = \begin{bmatrix} \mathbf{I}_n & \mathbf{I}_n \triangle t & \mathbf{I}_n \triangle t^2/2 \\ \mathbf{0}_n & \mathbf{I}_n & \mathbf{I}_n \triangle t \\ \mathbf{0}_n & -\dfrac{\mathbf{K}}{\mathbf{M}} \triangle t & \mathbf{I}_n - \dfrac{\mathbf{C}}{\mathbf{M}} \triangle t \end{bmatrix}$, $G = \begin{bmatrix} 0_{2n \times 1} \\ S_f \end{bmatrix}$, and

$$d_{k-1} = \frac{F_0 w_0 \triangle t}{m_1} \cos[w_0(k-1)\triangle t + \phi_0]. \tag{10}$$

Under the assumption that the dynamic and measurement systems are disturbed by white noise, we have the following state-space equations *without* direct feed-through of d_{k-1} into the measurements Y_k, given by:

$$\mathcal{X}_k = \mathbf{A}\mathcal{X}_{k-1} + G \mathrm{d}_{k-1} + Q_{k-1}, \tag{11}$$

$$Y_k = \mathbf{H}\mathcal{X}_k + R_k, \tag{12}$$

where $Q_{k-1} \sim \mathcal{N}(0, \mathbf{\Sigma_Q})$ is the Gaussian process noise with zero mean and a $3n \times 3n$ covariance matrix $\mathbf{\Sigma_Q}$, $\mathbf{H} = [\mathbf{0}_{n \times 2n}\ \mathbf{I}_n]$, and $R_k \sim \mathcal{N}(0, \mathbf{\Sigma_R})$ is the zero mean observation noise with an $n \times n$ covariance matrix $\mathbf{\Sigma_R}$. Notably, there is no d_{k-1} term in (12), thus, this system is called a linear system *without* direct feed-through.

The Minimum Variance Unbiased (MVU) Filter: Since $\operatorname{rank}(\mathbf{H}G) = 1$, and $\operatorname{rank}(G) = 1$, the *Assumption 1* in [34] holds. Therefore, we can apply the MVU filter as follows:

$$\begin{aligned}\hat{\mathcal{X}}_{k|k-1} &= \mathbf{A}\hat{\mathcal{X}}_{k-1}, \\ \mathbf{P}_{k|k-1} &= \mathbf{A}\mathbf{P}_{k-1}\mathbf{A}^T + \mathbf{\Sigma_Q}, \\ \tilde{\mathbf{R}}_k &= \mathbf{H}\mathbf{P}_{k|k-1}\mathbf{H}^T + \mathbf{\Sigma_R}, \\ V_k &= \mathbf{H}G = S_f, \\ \mathbf{F}_k &= \mathbf{P}_{k|k-1}\mathbf{H}, \\ \Pi_k &= (V_k^T \tilde{\mathbf{R}}_k^{-1} V_k)^{-1} V_k^T \tilde{\mathbf{R}}_k^{-1} = 1, && (13) \\ L_k &= G, && (14) \\ \mathbf{P}_k &= \mathbf{P}_{k|k-1} - G\tilde{\mathbf{R}}_k^{-1} G^T - F_k G^T - G F_k^T, \\ \hat{\mathcal{X}}_k &= \hat{\mathcal{X}}_{k|k-1} + L_k(Y_k - \mathbf{H}\hat{\mathcal{X}}_{k|k-1}), && (15) \\ \hat{d}_{k-1} &= \Pi_k(Y_k - \mathbf{H}\hat{\mathcal{X}}_{k|k-1}) = Y_k - \mathbf{H}\hat{\mathcal{X}}_{k|k-1}. && (16)\end{aligned}$$

Remark 1: The estimated feed-through $\hat{d}_{k-1}$ in (16) of d_{k-1} in (10) has the form of the derivative of the input force f (see (7) and (9)) since $\Pi_k = 1$ in (13) and $L_k = G$ in (14). Therefore, it requires an additional processing step to compute f. For example, we can use a fast Fourier transform (FFT) algorithm to extract the unknown amplitude $F_0 w_0$ and frequency w_0 of d_{k-1}. However, if w_0 is incorrectly estimated, the estimation error of the input amplitude F_0 compared to its ground truth can be substantially large.

IV. Estimating an unknown input force without any priors

In this section, we consider the problem of estimating the unknown input force without any prior, by formulating the problem as a linear system *with* direct feed-through of inputs to measurements. Since the input forces are unknown, we cannot take its derivative to apply the discretisation method that includes the acceleration as an element of a state as in the previous section. Instead, by selecting the state vector $\mathcal{X} = [X^T, \dot{X}^T]^T = [X_1^T, X_2^T]^T$, (1) can be discretised to obtain:

$$\begin{bmatrix} X_{1,k+1} \\ X_{2,k+1} \end{bmatrix} = \begin{bmatrix} \mathbf{I}_n & \mathbf{I}_n \triangle t \\ -\dfrac{\mathbf{K}}{\mathbf{M}} \triangle t & \mathbf{I}_n - \dfrac{\mathbf{C}}{\mathbf{M}} \triangle t \end{bmatrix} \begin{bmatrix} X_{1,k} \\ X_{2,k} \end{bmatrix} + \begin{bmatrix} 0_{n\times 1} \\ \mathbf{M}^{-1} S_f \triangle_t \end{bmatrix} f_k. \tag{17}$$

Assume that only accelerations $\ddot{X}$ are measured. Thus, based on (1) the noiseless measurement vector Y as follows:

$$Y_k = -[\frac{\mathbf{K}}{\mathbf{M}} \quad \frac{\mathbf{C}}{\mathbf{M}}] \times [X_{1,k}, X_{2,k}]^T + \mathbf{M}^{-1} S_f f_k \tag{18}$$

Assuming, as before, that the dynamic and measurement systems are disturbed by white noise, we have the following state-space equations:

$$\mathcal{X}_{k+1} = \mathbf{A}\mathcal{X}_k + B f_k + Q_k, \tag{19}$$

$$Y_k = \mathbf{G}\mathcal{X}_k + J f_k + R_k, \tag{20}$$

where $\mathbf{A} = \begin{bmatrix} \mathbf{I_n} & \mathbf{I_n}\triangle \mathbf{t} \\ -\dfrac{\mathbf{K}}{\mathbf{M}}\triangle t & \mathbf{I_n} - \dfrac{\mathbf{C}}{\mathbf{M}}\triangle \mathbf{t} \end{bmatrix}$, $B = \begin{bmatrix} 0_{n\times 1} \\ \mathbf{M}^{-1} S_f \triangle_t \end{bmatrix}$, $\mathbf{G} = -\begin{bmatrix} \dfrac{\mathbf{K}}{\mathbf{M}} & \dfrac{\mathbf{C}}{\mathbf{M}} \end{bmatrix}$, $J = \mathbf{M}^{-1} S_f$, $Q_{k-1} \sim \mathcal{N}(0, \mathbf{\Sigma_Q})$ is the Gaussian process noise with zero mean and a $2n \times 2n$ covariance matrix $\mathbf{\Sigma_Q}$, and $R_k \sim \mathcal{N}(0, \mathbf{\Sigma_R})$ is the zero mean observation noise with an $n \times n$ covariance matrix $\mathbf{\Sigma_R}$. Notably, we have the f_k term in both dynamic and measurement models, *i.e.*, (19) and (20), thus, this system is called a linear system *with* direct feed-through to measurements.

The Ensemble Kalman Filter Based on AKF (EnAKF): In this work, we derive the EnAKF filter by combining the ensemble sampling method from EnKF filter [42], [45] and the AKF filter [37] that augments the unknown force into the state, *i.e.*,

$$\mathcal{X}_k^a = [\mathcal{X}_k^T \quad f_k]^T. \tag{21}$$

Thus, the augmented state equation is obtained based on (19),(20):

$$\mathcal{X}_{k+1}^a = \mathbf{A}^a \mathcal{X}_k^a + Q_k^a, \tag{22}$$

$$Y_k = \mathbf{G}_a \mathcal{X}_k^a + R_k, \tag{23}$$

where $\mathbf{A}^a = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{0}_n & \mathbf{I}_n \end{bmatrix}$; $Q_k^a \sim \mathcal{N}(0, \mathbf{\Sigma_Q^a})$ is the Gaussian process noise with zero mean and a $(2n+1)\times(2n+1)$ covariance matrix $\mathbf{\Sigma_Q^a} = \begin{bmatrix} \mathbf{\Sigma_Q} & 0_{2n\times 1} \\ 0_{1\times 2n} & \Sigma_f \end{bmatrix}$; Σ_f is the initial estimation covariance noise of the input force f; $\mathbf{G}_a = [\mathbf{G} \quad J]^T$; $R_k \sim \mathcal{N}(0, \mathbf{\Sigma_R})$ is the zero mean observation noise with an $n \times n$ covariance matrix $\mathbf{\Sigma_R}$.

Suppose that at time k, the state $\mathcal{X}_k^a$ is approximated by an ensemble of q possible states $\mathcal{X}_k^e$, where

$$\mathcal{X}_k^e \approx \{\mathfrak{X}_k^{(1)}, \ldots, \mathfrak{X}_k^{(q)}\}. \tag{24}$$

Then the EnAKF filter can be described using the two following stages:

1. Forecast step:

$$\begin{aligned}
\hat{\mathfrak{X}}_k^{(i)} &= \mathbf{A}^a \mathfrak{X}_{k-1}^{(i)} + Q_k^{a,(i)}, \ \forall i = 1, \ldots, q, \\
\bar{\mathcal{X}}_k^e &= (\sum_{i=1}^{q} \mathfrak{X}_k^{(i)})/q, \\
Y_k^{(i)} &= \mathbf{G}^a \hat{\mathfrak{X}}_k^{(i)} + R_k^{(i)}, \ \forall i = 1, \ldots, q \qquad (25) \\
\bar{Y}_k &= (\sum_{i=1}^{q} Y_k^{(i)})/q, \\
\mathbf{E}_k^x &= [\hat{\mathfrak{X}}_k^1 - \bar{\mathcal{X}}_k^e, \ldots, \hat{\mathfrak{X}}_k^q - \bar{\mathcal{X}}_k^e], \\
\mathbf{E}_k^y &= [Y_k^1 - \bar{Y}_k, \ldots, Y_k^q - \bar{Y}_k], \\
\mathbf{P}_k^{xy} &= \mathbf{E}_k^x (\mathbf{E}_k^y)^T/(q-1), \qquad (26) \\
\mathbf{P}_k^{yy} &= \mathbf{E}_k^y (\mathbf{E}_k^y)^T/(q-1). \qquad (27)
\end{aligned}$$

2. Analysis step:

$$\mathbf{K}_k = \mathbf{P}_k^{xy} (\mathbf{P}_k^{yy})^{-1}, \tag{28}$$

$$\mathfrak{X}_k^{(i)} = \hat{\mathfrak{X}}_k^{(i)} + \mathbf{K}_k (Y_k + R_k^{(i)} - \mathbf{G}^a \hat{\mathfrak{X}}_k^{(i)}), \ \forall i = 1, \ldots, q \tag{29}$$

where $\mathbf{A}^a$ and $\mathbf{G}^a$ are defined in (22) and (23), while $Q_k^{a,(i)}$ and $R_k^{(i)}$ are an ensemble of Q_k^a and R_k, respectively.
Remark 2: The Kalman Gain $\mathbf{K}_k$ in (28) is calculated via the covariance error $\mathbf{P}_k^{xy}$ in (26) and $\mathbf{P}_k^{yy}$ in (27) instead of the full covariance matrix of $\mathbf{P}_k^{xx}$ with much higher dimensions. Therefore, the EnAKF filter leads to substantially reduced the computational cost.

V. Experiments

In this section, we conduct two experiments to validate and demonstrate the effectiveness of the proposed algorithm. First, a comprehensive study of applying the MVU filter to estimate an unknown input force with a sinusoidal form is investigated. Second, we compare our proposed EnAKF and other state-of-the-art methods, including AKF, DFK and GDF filters when estimating the unknown input force without any prior. All of the numerical experiments were conducted on a desktop computer with an Intel(R) Core(TM) i7-6700 CPU @ 3.4 GHz with 32 GB RAM and using MATLAB-R2019b software.

A. Experiment 1 — *without* direct feed-through formulation

We estimate the force parameters using a minimum-variance unbiased (MVU) filter and a fast Fourier transform (FFT) for a 100-DoF system. In this example, we implement the minimum-variance unbiased (MVU) filter to estimate the unknown d_k signal in a 100-DoF system and subsequently use an FFT to estimate the unknown force parameters: i) amplitude, and ii) frequency.

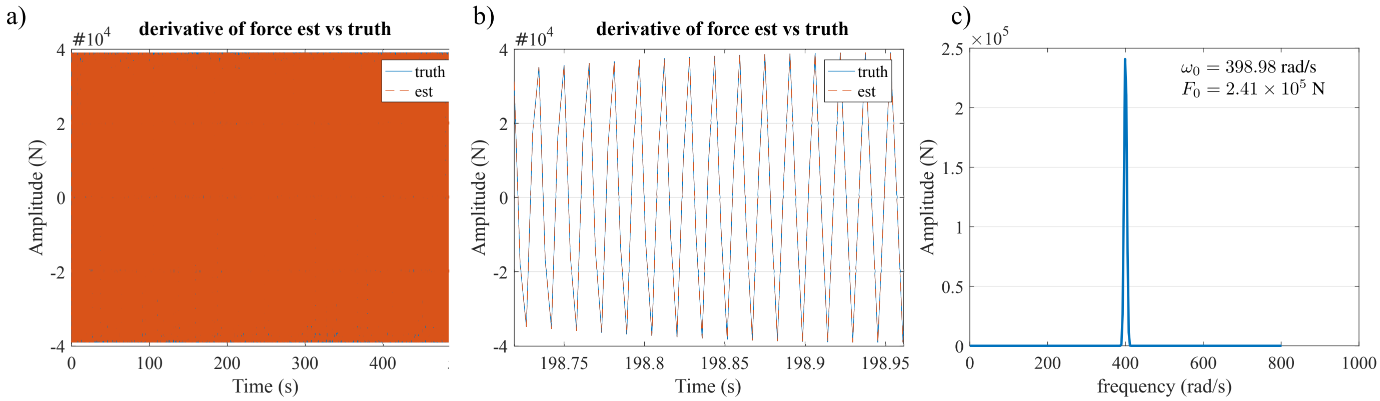

Fig. 2. Estimated values of $\hat{d}_k = \dfrac{F_0\omega_0\triangle t}{m_1}\cos(\omega_0 k\triangle t + \phi_0)$ versus its ground truth: a) over 500 s; b) over a small window for clarity. c) Estimated force parameters based on $\hat{d}_k$'s values.

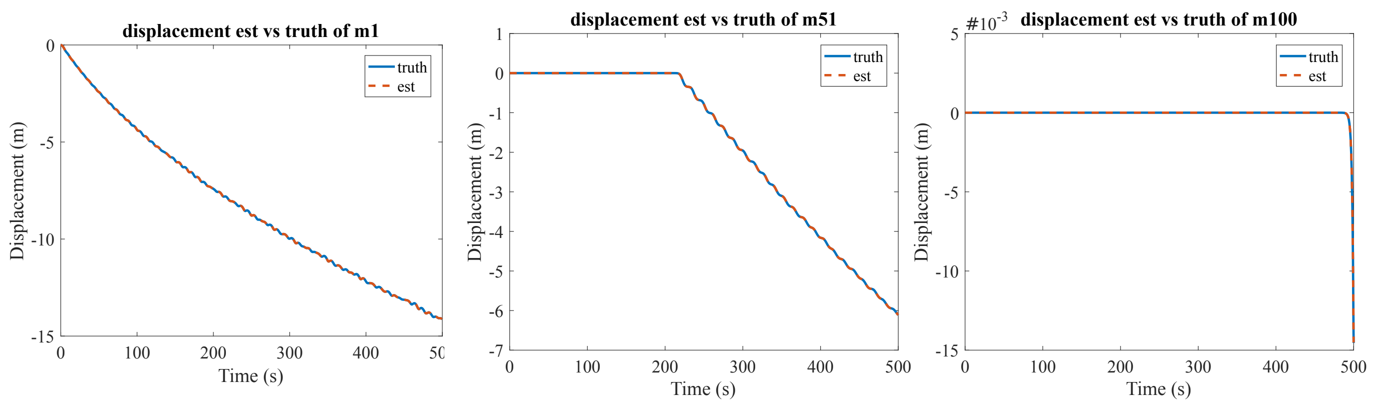

Fig. 3. Comparison of estimated and truth displacement values.

Parameter values of the 100-DoF system are: $M = \text{diag}([\text{linspace}(10, 100, 100)])$ kg; $[c_1, \ldots, c_{100}] = \text{linspace}(2.5, 10, 100)$ Ns/m; $[\tilde{c}_1, \ldots, \tilde{c}_{100}] = 100 \cdot \text{linspace}(2.5, 10, 100)$ Ns/m; $[k_1, \ldots, k_{100}] = \text{linspace}(3, 10, 100) \times 10^5$ N/m; $[\tilde{k}_1, \ldots, \tilde{k}_{100}] = 2 \cdot 10^7 \cdot \text{linspace}(3, 10, 100)$ N/m. Here, $\text{linspace}(x_1, x_2, N)$ is a MATLAB function that generates N points distributed equally between x_1 and x_2. An external force with amplitude $F_0 = 2.5 \cdot 10^5$ N, angular frequency $\omega_0 = 400$ rad/s, and initial phase $\phi_0 = \pi/6$ rad are considered for the simulation.

In this experiment, we set the sampling time step $\triangle t = 0.0039\ s$ with a sensor sampling rate of $F_s = 1/\triangle t = 256\ Hz$, while total experiment time is set at 500 s. Since F_0 and ω_0 is not part of the state vector, the filter does not need to assume any prior information about these parameters. For FFT parameters, we set a number of FFT bins $N_{FFT} = 512$ samples, with a total measurement time equal to $N_{FFT}/F_s = 2$ s, a 4-term Blackman-Harris window is used as a windowing function.

Fig. 2a and b show the estimated values of the derivative of the input force $\hat{d}_k$ versus its ground truth d_k using the MVU filter. Fig. 2c shows the estimated force parameters, frequency w_0 and amplitude F_0, obtained using an FFT based on $\hat{d}_k$ values. The results demonstrate that this method is able to estimate the force parameters accurately with total delay in this case of 2 s or 512 $samples$.

Fig. 3 shows the comparison between estimated and ground-truth values of displacement for a few representative masses in the 100-DoF system over 500 s under the influence of the external force. As expected, the results confirm that when applying an external force continuously, displacements occur across all the masses in the 100-DoF system, over time.

The overall estimated error of input force parameters and the displacement of the first contact point at mass m_1 is reported in Table. I. The results demonstrate that the MVU filter can accurately estimate the force-frequency ω_0 with the estimated error of only 0.25 %. Although the absolute error of the force amplitude F_0 is high, its relative error is less than 5 %. Since the MVU filter is an unbiased and optimal filter, its estimated displacement error is negligible. As discussed in Remark 1, if the estimation error of w_0 is large, then the estimated value of the input amplitude F_0 can be considerably different compared to its ground truth. However, as observed in this experiment, the FFT algorithm accurately calculates the estimated frequency of w_0 with a small relative error at 0.25%.

TABLE I
ESTIMATED ERRORS OF INPUT FORCE PARAMETERS AND THE DISPLACEMENT OF m_1

	F_0 (N)	ω_0 (rad)	Displacement error of m_1(m)
Absolute Error	9.1e3	1.02	1.1e-4
Relative Error (%)	3.64	0.25	0.014

As a result, F_0 is estimated correctly using our proposed MVU filter with the derivative approach for input forces.

The processing time for this MVU filter of a 100-DoF system over the total experiment time of 500 s is 400.50 s; hence, it is feasible to estimate the unknown input force parameter values, even for a complex 100-DoF system, in a real-time manner.

*B. Experiment 2 — **with** direct feed-through formulation*

We compare the performance of GDF, AKF, DKF and EnAKF filters. In this example, we consider the same mass-spring-damper system and force parameters as in **Example 1** with a 100-DoF system; notably, in Example 1, the force parameters were subsequently estimated using an FFT. As remarked in [40], GDF does not require any prior information or assumptions on the unknown input. In contrast, the AKF, DKF, and EnAKF filters require an initial estimate for its input mean and covariance at time 0. The number of ensembles used for EnAKF is $q = 500$, while the total measurement time used in the experiments is 5 s.

Fig. 4 a and b show the estimated values of input force $\hat{f}_k$ versus its ground truth f_k using GDF, AKF, DKF and EnAKF. The results show that all four methods can accurately estimate the unknown input force values without knowing its form, *i.e.*, the sinusoidal form in our case.

Fig. 4c shows the estimated force parameter results based on $\hat{f}_k$ values using a 4-term Blackman-Harris window and an FFT operation to extract the frequency w_0 and amplitude F_0 of the input force. The results demonstrate that these methods can estimate the force parameters accurately with a total delay of, in this case, 2 s or 512 samples (since the four methods result in the same estimated parameters, only one graph is plotted here).

Fig. 5 depicts the estimated displacements of different filtering methods at masses m_1, m_{51} and m_{100}. Further, Table II shows the estimated errors of input force parameters and the displacement of m_1—the first mass in contact with the input force. The results show that all of the filtering methods provide the same estimated results for both amplitude F_0 and frequency w_0 of the input force. However, for displacement estimation, GDF outperforms the other filters, while EnAKF shows relatively poor performance. The reason is that GDF is the optimal filter, while the remaining filters are only sub-optimal. Further, the EnAKF filter is the approximated filter of the AKF filter. Thus its estimation performance can be expected to be slightly degraded compared to the AKF filter.

Table. III depicts a comparison of the processing times for the different filters for various degrees-of-freedom n_{dof}. The results demonstrate that when the degrees-of-freedom is small, the AKF filter consistently outperforms other filters in term of computational time. However, when n_{dof} is large, *i.e.*, $n_{dof} \geq 100$, the EnAKF filter requires the least computational power since it does not need to compute the full covariance error matrix.

TABLE II
ESTIMATION ERRORS OF INPUT FORCE PARAMETERS AND THE DISPLACEMENT OF m_1

Filters	F_0(%)	ω_0(%)	Displacement error of m_1(m)
MVU*	3.64	0.25	0.000
DKF	3.89	0.25	0.006
GDF	3.89	0.25	0.000
AKF	3.89	0.25	0.044
EnAKF	3.89	0.25	0.099

TABLE III
COMPARISON OF PROCESSING TIMES FOR DIFFERENT FILTERS VERSUS THE DEGREES-OF-FREEDOM NUMBER

n_{dof}	*10*	*20*	*30*	*50*	*100*
MVU* (s)	0.24	0.44	0.66	1.48	5.11
DKF (s)	**0.60**	1.28	3.31	4.92	20.72
GDF (s)	0.99	2.14	6.24	9.12	39.15
AKF (s)	0.61	**1.02**	**2.51**	**3.36**	15.05
EnAKF (s)	1.42	2.29	4.09	4.88	**13.16**

* Note: The MVU filter assumes the force to be unknown but modelled and differentiable. According to MVU's derivation, the covariance error matrix P_k is updated without calculating the inversion of its predicted value $\hat{P}_k$. Thus its computational cost is smaller compared to the other filters. Since the MVU filter is categorised as *without* direct feed-through while the remaining filters are *with* direct feed-through types, its computational time is provided for completeness and as a baseline for comparing among *direct* feed-through filters where a model for the force is not required.

Discussion: Although the EnAKF filter does not perform well in the estimation of the displacements, its performance in estimating the unknown and unmodelled input forces is comparable to other filters and is the fastest filter when n_{dof} is large. On the other hand, the GDF filter is the best performing filter in terms of estimating the displacement; however, the GDF filter takes a longer time to compute. Furthermore, as observed in our numerical experiments and in [3], the GDF filter often suffers from numerical instability caused by the failure to evaluate the inversion of the error covariance matrix $\mathbf{P}_k^f = (J^T \tilde{\mathbf{R}}_k^{-1} J)^{-1}$ if the measurement period is long enough. The reason is that GDF is subject to spurious low-frequency drifts caused by acceleration's insensitivity to any quasi-static components in the input force [26], wherein dummy displacement measurements can be used to mitigate the issues [39]. Hence, we can only compare the four filters within 5 seconds of measurements instead of 500 seconds as in the case of the MVU filter presented in Sec V-A.

Further, as presented in Table. II and Table. III, the MVU filter is the best performing filter across all measured metrics. Hence, if the unknown input force is differentiable and can be modelled in advance, the MVU filter provides an optimal method of estimating the input force.

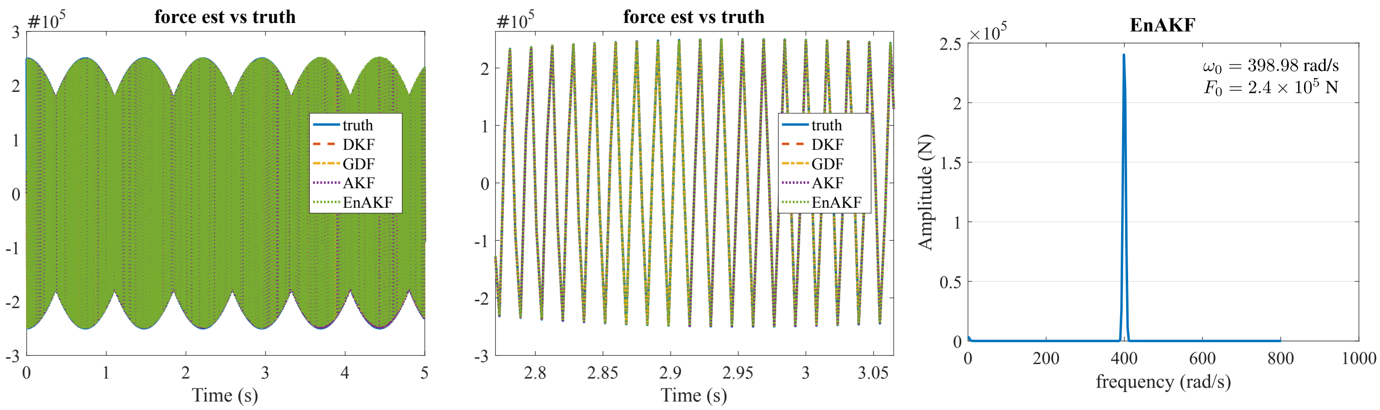

Fig. 4. Estimated values of f based on GDF, AKF, DKF and EnAKF versus its ground truth: a) over 5 s; b) over a small period for clarity. c) Force parameters estimated from $\hat{f}$ values (since four methods result in the same estimated parameters, only one graph is plotted here).

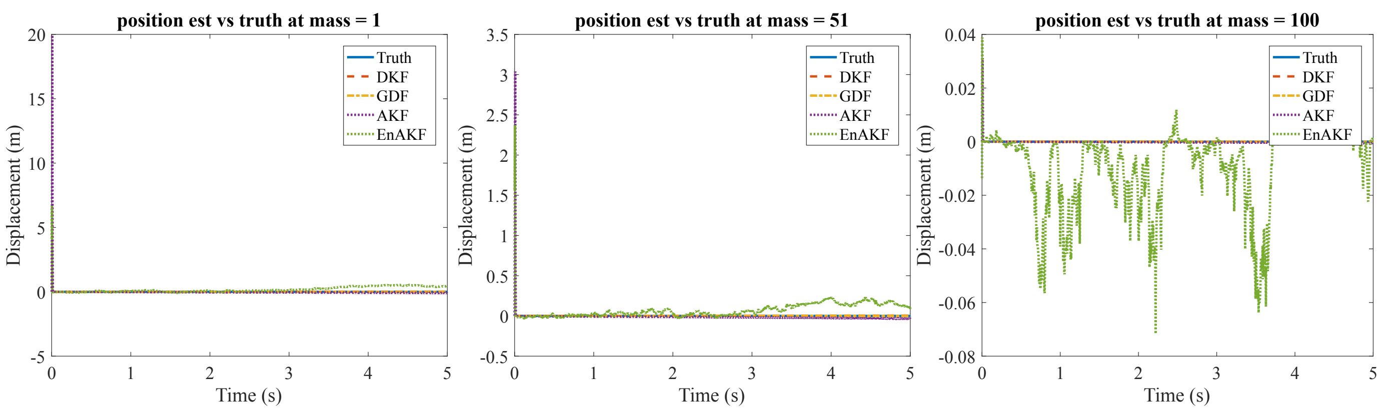

Fig. 5. Displacement estimations based on GDF, AKF, DKF and EnAKF versus truth values of a set of representative masses.

VI. Conclusion

In this paper, we have proposed two novel approaches for estimating an unknown input force applied to a large structural system. If the input force has a sinusoidal form, the MVU filter coupled with a force extraction technique from estimated force derivative can be implemented to estimate the unknown force in real-time environments for the large structural system. When the input force is completely unknown, our proposed EnAKF filter can significantly reduce computational time compared to other state-of-the-art methods without degrading the estimation performance.

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